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Noncoercive Hemivariational Inequality Approach to Constrained Problems for Star-Shaped Admissible Sets

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Abstract. The hemivariational inequality approach is used in order to establish the existence of solutions to a large class of noncoercive constrained problems in a reflexive Banach space, in which the set of all admissible elements is not convex but fulfills some star-shaped property.

Key words: Noncoercive hemivariational inequalities, pseudomonotone operators, star-shaped admissible sets, nonconvex unilateral problems in elasticity.

1. Introduction

This paper follows the work of Naniewicz [15] concerning the study of constrained problems in a reflexive Banach space, in which the set of all admissible elements is not convex but fulfills some star-shaped property.

In [15] Naniewicz proved some remarkable existence results for the following problem:

Problem P. Find $u \in C$ such that
$$\langle Au - f, v\rangle \geq 0, \ \forall v \in T_C(u),$$

where the set C is assumed to be closed and star-shaped with respect to a certain ball, $T_C(u)$ denotes Clarke's tangent cone of C at $u \in C$, A is assumed to be a pseudomonotone and coercive operator and f is given in X^*. However, the variational formulation of some engineering problems leads to hemivariational inequalities which are noncoercive. For instance, the lack of coercivity may be due to boundary conditions which are insufficiently blocking-up. By using the recession approach developed by Adly, Goeleven and Théra [1], we are able to extend the theory of Naniewicz to problem P when A is no more coercive.

Problem P is a special case of a great class of problems called hemivariational inequalities which have been introduced by Panagiotopoulos [20] in order to formulate various mechanical problems connected to energy functionals which are

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neither convex nor differentiable. As a consequence of the contributions of Panagiotopoulos, the study of hemivariational inequalities has emerged as an interesting branch of applied mathematics and this topic is now the subject of the attention of several engineers and mathematicians (see Goeleven and Théra [13], Naniewicz [15]–[17] and Panagiotopoulos [18]–[22]).

For details concerning the possible applications of this kind of model, we refer the reader to the books of Naniewicz and Panagiotopoulos [17], and Panagiotopoulos [21], [22].

The hemivariational inequality considered in this paper permits the formulation of new problems in the theory of elasticity. Indeed, problem P is connected with the treatment of linear elastic bodies whose displacement field is subjected to constraints expressed by means of nonconvex star-shaped and closed admissible sets. These problems have been studied for the first time in the book of Naniewicz and Panagiotopoulos [17]. However, the approach examined in [17] requires blocking-up boundary conditions in order to guarantee the coercivity of the underlying operator A. The study of the same problems with insufficiently blocking-up boundary conditions becomes now possible by the theory developed in this paper.

2. Preliminaries, Notations and Basic Facts

Let X be a real reflexive Banach space. Let us denote further by X^* the dual space to X and let $\langle .,. \rangle$ be the duality pairing between X and X^*. The norm in X is denoted by $\| \, . \, \|$ and on X^* by $\| \, . \, \|_*$. We will write " $\rightarrow$ " and " $\rightharpoonup$ " to denote respectively the strong convergence and the weak convergence. For a nonempty subset D of X, we write $int\{D\}$ for the interior of D in X and $cl(D)$ for the closure. For an operator $A : X \rightarrow X^*$, we write $Ker(A)$ for the kernel, $R(A)$ for the range and $D(A)$ for the domain.

An operator $T : X \rightarrow 2^{X^*}$ is said to be *pseudomonotone* (see Browder and Hess [6]) if

(i) the set Tu is nonempty, bounded, closed and convex for any $u \in X$;

(ii) T is upper semicontinuous from each finite dimensional subspace F of X to X^* equipped with the weak topology, i.e. to a given element $f \in F$ and a weak neighborhood V of $T(f)$ in X^* there exists a neighborhood U of f in F such that $T(u) \subset F$ for all $u \in U$;

(iii) if $u_n \rightharpoonup u$ and if $z_n \in T(u_n)$ is such that

$$\limsup \langle z_n, u_n - u \rangle \leq 0,$$

then for each $v \in V$ there exists $z(v) \in T(u)$ such that

$$\liminf \langle z_n, u_n - v \rangle \geq \langle z(v), u - v \rangle.$$

An operator $A : X \to X^*$ is said to have the S^+ *– property* (see Pascali and Sburlan [23]) if $u_n \rightharpoonup u$ and $\limsup\langle Au_n, u_n - u\rangle \leq 0$ implies that $u_n \to u$.

Let $G : X \to I\!R \cup \{+\infty\}$ be a functional. Then its behavior at infinity can be described in terms of what is called the *recession function of G* (see Baiocchi *et al.* [4]), defined as follows

$$G_\infty(x) : = \liminf_{\substack{t\to+\infty\\ v\to x}} G(tv)/t$$
$$= \inf\{\liminf_{n\to\infty} G(t_n v_n)/t_n : t_n \to \infty, v_n \to x\}.$$

We recall the basic properties of the recession function. Let G, H be two functionals defined on X with values in $(-\infty, +\infty]$. Then

P$_1$. G_∞ is lower semicontinuous and positively homogeneous of degree 1;

P$_2$. $(G + H)_\infty \geq G_\infty + H_\infty$;

P$_3$. If H is positively homogeneous of order 1 and continuous, then

$$(G + H)_\infty = G_\infty + H;$$

P$_4$. If G is non-negative, positively homogeneous of degree greater than 1 and lower semicontinuous, then

$$G_\infty(x) = \begin{cases} +\infty \text{ if } G(x) \neq 0 \\ 0 \text{ if } G(x) = 0. \end{cases}$$

Let $u_o \in X$, we intoduce the concept of *recession function associated to a general operator* $A : X \to X^*$ *with respect to* u_o by the formula

$$\underline{r}_{uo,A}(x) : = \liminf_{\substack{t\to\infty\\ v\to x}} \langle A(tv), tv - u_o\rangle/t.$$

If we set $G(x) := \langle Ax, x - u_o\rangle$, then clearly

$$\underline{r}_{uo,A}(x) = G_\infty(x).$$

If $u_o = 0$, then our definition reduces to the one introduced by Brézis and Nirenberg [5] in order to characterize the range of some nonlinear operators.

Let K be a subset of X, the *recession cone of K* is the closed cone

$$K_\infty := dom[\{\psi_K\}_\infty] = \{x \in X : \{\psi_K\}_\infty(x) < +\infty\},$$

where ψ_K denotes the indicator function of K. Equivalently, this amounts to say that x belongs to K_∞ if and only if there exists sequences $\{t_n \mid n \in I\!N\}$ and $\{x_n \mid n \in I\!N\} \subset K$ such that $t_n \to +\infty$ and $t_n^{-1} x_n \to x$.

Let $C \subset X$ be a nonempty closed subset. We denote by

$$T_C(u) : = \{k \in X : \forall u_n \in C, u_n \to u, \forall \lambda_n \downarrow 0, \\ \exists k_n \to k : u_n + \lambda_n k_n \in C\},$$

the *Clarke's tangent cone* of C at u, by

$$N_C(u) := \{u^* \in X^* : \langle u^*, k\rangle \ \leq \ 0, \forall k \in T_C(u)\},$$

the *Clarke's normal cone* to C at u, by

$$d_C(u) := \inf_{w \in C} \parallel u - w \parallel,$$

the *distance function* of C, by

$$d_C^0(u,v) : = \limsup_{y \to u, t \downarrow 0} [d_C(y + tv) - d_C(y)]/t,$$

the *generalized directional derivative of* d_C *at* u *in the direction* v and by

$$\partial d_C(u) := \{w \in X^* : d_C^0(u,v) \ \geq \ \langle w, v\rangle, \forall v \in X\},$$

the Clarke's *generalized gradient of* d_C *at* u (see Clarke [8]).

Let $B(u_0, \rho)$ be a closed ball in X with center u_o and radius $\rho > 0$. We say that C is *star-shaped* with respect to $B(u_o, \rho)$ (see Naniewicz [15]) if

$$v \in C \Leftrightarrow \lambda v + (1 - \lambda)w \in C, \ \forall \lambda \in [0,1], \quad \forall w \in B(u_o, \rho).$$

We resume in the following lemma the basic results concerning the function distance of a star-shaped set which have been proved by Naniewicz.

LEMMA 2.1. (Naniewicz; [15]). *Let X be a real reflexive Banach space, C a nonempty closed subset of X. If C is star-shaped with respect to $B(u_o, \rho)$ then*

(1) $d_C^0(u, u_o - u) \ \leq \ -d_C(u) - \rho, \ \forall u \notin C$

(2) $d_C^0(u, u_o - u) = 0, \ \forall u \in C.$

The following result concerning the pseudomonotonicity property of the generalized Clarke's gradient is also due to Naniewicz [16].

LEMMA 2.2. (Naniewicz; [16]). *Let X be a real reflexive Banach space. Let $f_i : X \to \mathbb{R}$ be a finite collection of locally Lipschitzian convex functions defined on X. Define $f : X \to \mathbb{R}$ as*

$$f(u) := \min\{f_i(u) : i = 1, ...N\}, \ u \in X.$$

Let $A : X \to X^*$ be a maximal monotone operator with $D(A) = X$ and satisfying the S^+-property. Then $A + \partial f$ is pseudomonotone.

As a direct consequence of this Lemma, we get the following result.

PROPOSITION 2.1. *Let X be a real reflexive Banach space, $A : X \to X^*$ a maximal monotone operator with $D(A) = X$ and satisfying the S^+-property. Let C be a subset of X which can be represented as the union of a finite collection of nonempty closed convex subsets $C_j (j = 1, ..., N)$ of X, i.e. $C = \cup_{j=1}^N C_j$. We assume that $int(\cap_{j=1}^N C_j) \neq \emptyset$. Then (i) C is star-shaped with respect to a certain ball and (ii) for each $\lambda \geq 0, A + \lambda\partial d_C$ is pseudomonotone.*

Proof. (i) trivial. (ii) The distance function of C is expressed as a pointwise minimum of the Lipschitzian convex functions $d_i : X \to \mathbb{R}$, where d_i denotes the distance function of C_i, and the result follows from Lemma 2.2. ∎

3. Constrained Problems

Let us introduce the following set of asymptotic directions (cf. Adly, Goeleven and Théra [1] and Tomarelli [25]):

$$R(A, f, u_o) := \{w \in X : \exists u_n \in X, \| u_n \| \to +\infty, w_n := u_n \| u_n \|^{-1} \rightharpoonup w \text{ and } \langle Au_n, u_n - u_o\rangle \leq \langle f, u_n - u_o\rangle\}.$$

DEFINITION 2.1. We say that $R(A, f, u_o)$ is *a-compact* if the following property holds true: If $\{w_n \mid n \in \mathbb{N}\}$ is a sequence such that

$$w_n := \frac{u_n}{\| u_n \|} \rightharpoonup w,$$

with

$$\langle Au_n, u_n - u_0\rangle \leq \langle f, u_n - u_0\rangle$$

and

$$\| u_n \| \to +\infty,$$

then $w_n \to w$.

Several examples of operators for which $R(A, f, u_o)$ is a-compact can be found in Adly *et al.* [1]. Further properties of $R(A, f, u_o)$ are contained in the following three propositions.

PROPOSITION 3.1. *Let u_o be given in X and f in X^*. If*

(i) *A satisfies the S^+-property;*

(ii) $\langle Ax, x\rangle \geq 0, \forall x \in X$;

(iii) *A is weakly continuous, i.e. $x_n \rightharpoonup x \Rightarrow Ax_n \rightharpoonup Ax$;*

(iv) *A is positively homogeneous.*

Then $R(A, f, u_o)$ is a-compact and

$$R(A, f, u_o) \subset \{w \in X \backslash \{0\} : \langle Aw, w \rangle = 0\}.$$

Proof. Let $w \in R(A, f, u_o)$. There exists $u_n \in X$ such that $t_n := \| u_n \| \to +\infty, w_n := u_n/t_n \rightharpoonup w$, and

$$\langle Au_n, u_n - u_o \rangle \ \leq \ \langle f, u_n - u_o \rangle \tag{1}$$

Dividing (1) by t_n^2, we obtain

$$\langle Aw_n, w_n \rangle \ \leq \ \langle Aw_n, \frac{u_o}{t_n} \rangle + \langle \frac{f}{t_n}, w_n - \frac{u_o}{t_n} \rangle \tag{2}$$

and thus, by assumption (iii),

$$\limsup \langle Aw_n, w_n \rangle \ \leq \ 0. \tag{3}$$

We have

$$\limsup \langle Aw_n, w_n - w \rangle \ \leq \ \limsup \langle Aw_n, w_n \rangle + \limsup \langle Aw_n, -w \rangle,$$

so that, by assumption (ii) and (iii)

$$\limsup \langle Aw_n, w_n - w \rangle \ \leq \ \limsup \langle Aw_n, w_n \rangle.$$

This together with (3) imply that

$$\limsup \langle Aw_n, w_n - w \rangle \ \leq \ 0,$$

and thus, by assumption (i), the sequence w_n is strongly convergent to w, which proves the a-compacity of $R(A, f, u_o)$. Since $\| w_n \| = 1$ and $w_n \to w$, we have $\| w \| = 1$. Then using (2) again, we get $\langle Aw, w \rangle \ \leq \ 0$. Therefore, using assumption (ii) again, we obtain

$$R(A, f, u_o) \subset \{w \in X \backslash \{0\} : \langle Aw, w \rangle = 0\}.$$ ■

EXAMPLE 3.1. Set $X := H^1(\Omega)$, where Ω is an open bounded subset of class C^1 in $\mathbb{R}^n (n \ \geq \ 1, n \in \mathbb{N})$. Let $A : X \to X^*$ be the bounded linear operator defined by

$$\langle Au, v \rangle := \textstyle\int_\Omega \nabla u . \nabla v \mathrm{d}x, \quad \forall\, u, v \in X.$$

It is easy to see that assumptions (ii)-(iv) are satisfied. It remains to prove that A satisfies the S^+-property. Indeed, let $\{w_n \mid n \in \mathbb{N}\}$ be a sequence such that $w_n \rightharpoonup w$ in $H^1(\Omega)$ (which implies that $w_n \to w$ in $L^2(\Omega)$ and $\nabla w_n \rightharpoonup \nabla w$ in $L^2(\Omega)$) and

$$\limsup \textstyle\int_\Omega \nabla w_n . \nabla (w_n - w) \mathrm{d}x \ \leq \ 0.$$

We get

$$\limsup \int_\Omega |\nabla w_n|^2\, dx \leq \limsup \int_\Omega \nabla w_n.\nabla(w_n - w) dx + \limsup \int_\Omega \nabla w_n.\nabla w dx$$

$$\leq \int_\Omega |\nabla w|^2\, dx.$$

Thus by the weak lower semicontinuity of the map $x \to \langle Ax, x\rangle$, we obtain

$$\int_\Omega |\nabla w|^2\, dx \leq \liminf \int_\Omega |\nabla w_n|^2\, dx \leq \limsup \int_\Omega |\nabla w_n|^2\, dx \leq \int_\Omega |\nabla w|^2\, dx.$$

Thus

$$\| \nabla w_n \|_{L^2} \to \| \nabla w \|_{L^2} .$$

Since $\nabla w_n \rightharpoonup \nabla w$ and the norm in $L^2(\Omega)$ is Kadec, we obtain

$$\nabla w_n \to \nabla w \text{ in } L^2(\Omega). \tag{4}$$

Using the fact that $w_n \to w$ in $L^2(\Omega)$ together with (4), we get

$$w_n \to w \text{ in } H^1(\Omega).$$ ∎

PROPOSITION 3.2. *Let u_o be given in X and f in X^*. If*

(i) *$R(A, f, u_o)$ is a-compact;*

(ii) *there exists a nonempty subset W of $X \backslash \{0\}$ such that*

$$R(A, f, u_o) \subset W$$

and

$$(C)\ \underline{r}_{uo,A}(w) > \langle f, w\rangle, \ \forall w \in W.$$

Then $R(A, f, u_o)$ is empty.

Proof. Suppose by contradiction that $R(A, f, u_o)$ is nonempty. Since $R(A, f, u_o) \subset W$ we have

$$\underline{r}_{uo,A}(w) > \langle f, w\rangle, \ \forall w \in R(A, f, u_o).$$

We can also find a sequence $\{u_n \mid n \in \mathbb{N}\}$ such that $t_n := \| u_n \| \to \infty$, $w_n := u_n / \| u_n \| \rightharpoonup w$
and

$$\langle A(t_n w_n), t_n w_n - u_o\rangle \leq \langle f, t_n w_n - u_o\rangle \tag{5}$$

Dividing (5) by t_n, we get

$$\langle A(t_n w_n)t_n^{-1}, t_n w_n - u_o\rangle \leq \langle f, w_n - \tfrac{u_o}{t_n}\rangle.$$

By assumption (i), $w_n \to w$ and thus

$$\underline{r}_{uo,A}(w) \leq \langle f, w\rangle,$$

a contradiction. ■

PROPOSITION 3.3. *Let X be a real reflexive Banach space such that $X \hookrightarrow L^2(\Omega)$ continuously (Ω denotes an open set in $\mathbb{R}^n$). Let u_o be given in X and f in X^*. We assume that $f(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in x and continuous in u. Assume for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$,*

$$(f_1) \quad | f(x,u) | \leq a | u | + b(x), a \geq 0, b \in L^2(\Omega)$$
$$(f_2) \quad u.f(x,u) \geq -c(x) | u | - d(x), c \in L^2(\Omega), d \in L^1(\Omega).$$

Let $A : X \to X^*$ be an operator satisfying assumptions (i)-(iv) of Proposition 3.1 and let $B : X \to X^*$ be the operator defined by

$$\langle Bu, v\rangle := \int_\Omega f(x,u).v\mathrm{d}x, \ \forall u, v \in X.$$

Then $R(A+B, f, u_o)$ is a-compact and

$$R(A+B, f, u_o) \subset \{w \in X\backslash\{0\} : \langle Aw, w\rangle = 0\}.$$

Proof. Let $w \in R(A, f, u_o)$. There exists $u_n \in X$ such that $t_n := \| u_n \| \to +\infty, w_n := u_n/t_n \rightharpoonup w$ and

$$\langle Aw_n, w_n\rangle + \langle t_n^{-1}Bu_n, w_n\rangle \leq \langle Aw_n, \tfrac{u_o}{t_n}\rangle + \langle t_n^{-2}Bu_n, u_o\rangle + \langle \tfrac{f}{t_n}, w_n - \tfrac{u_o}{t_n}\rangle$$

which means that

$$-\int_\Omega \tfrac{c(x)}{t_n^2} | u_n | \mathrm{d}x - \int_\Omega \tfrac{d(x)}{t_n^2}\mathrm{d}x + \langle Aw_n, w_n\rangle \leq \langle Aw_n, \tfrac{u_o}{t_n}\rangle + a\int_\Omega \tfrac{|u_n|}{t_n^2} | u_o | \mathrm{d}x + \int_\Omega \tfrac{b(x)}{t_n^2} | u_o | \mathrm{d}x + \langle \tfrac{f}{t_n}, w_n - \tfrac{u_o}{t_n}\rangle.$$

The embedding $X \hookrightarrow L^2(\Omega)$ is continuous and there exists $C > 0$ such that

$$\| u \|_{L^2} \leq C \| u \|, \ \forall u \in X.$$

Then it is easy to see that

$$\langle Aw_n, w_n\rangle \leq \langle Aw_n, \tfrac{u_o}{t_n}\rangle + \langle \tfrac{f}{t_n}, w_n - \tfrac{u_o}{t_n}\rangle + \alpha/t_n + \beta/t_n^2,$$

where

$$\alpha = C(\| c \|_{L^2} + a \| u_o \|_{L^2})$$

and

$$\beta = \| d \|_{L^1} + \| b \|_{L^2} \| u_o \|_{L^2} .$$

Thus

$$\limsup \langle A w_n, w_n \rangle \leq 0,$$

and we conclude as in Proposition 3.1 ■

Let us now come back to problem P. As in the article of Naniewicz [15] we will use a penalization method, but to prove the existence of a solution for the penalized problem we need another approach than the one used by Naniewicz [15] which is only valid for coercive problems.

For any $\lambda > 0$, we formulate the penalization problem.

Problem$(\mathbf{Q}_\lambda)$. Find $u_\lambda \in X$ such that :
$$\langle Au_\lambda - f, v - u_\lambda \rangle + \lambda d_C^0(u_\lambda, v - u_\lambda) \geq 0, \forall v \in X.$$

Suppose that the following hypotheses hold true.

(H$_1$) X is a real reflexive Banach space and C is a nonempty closed subset of X which is star-shaped with respect to a ball $B(u_o, \rho), \rho > 0$;

(H$_2$) $A + \lambda \partial d_C$ is pseudomonotone for each $\lambda > 0$;

(H$_3$) A is bounded.

LEMMA 3.1. *Suppose that assumptions* (H$_1$)-(H$_3$) *are satisfied. If*

$$R(A, f, u_o) = \emptyset$$

then problem (Q_λ) *has at least one solution.*

Proof. Let B_n be the nonempty closed bounded set defined by

$$B_n := \{x \in X : \| x \| \leq n\}.$$

Since d_C is Lipschitz continuous, the operator $\partial d_C : X \to X^*$ acts as a bounded operator, so that with assumptions (H$_1$)–(H$_3$) and by ([7]; theorem 7.8) (since B_n is bounded, the coercivity assumption of theorem 7.8 given in the book of Browder [7] is not necessary) there exist

$$u_{\lambda,n} \in B_n$$

and

$$z_{\lambda,n} \in (A + \lambda \partial d_C)(u_{\lambda,n})$$

such that

$$\langle z_{\lambda,n} - f, v - u_{\lambda,n} \rangle \ \geq \ 0, \ \forall v \in B_n.$$

Thus, we get

$$\lambda d_C^0(u_{\lambda,n}, v - u_{\lambda,n}) + \langle Au_{\lambda,n} - f, v - u_{\lambda,n} \rangle \ \geq \ 0, \ \forall v \in B_n. \tag{6}$$

We prove that there exists $k \in \mathbb{N}\backslash\{0\}$ such that $\| u_{\lambda,k} \| < k$. If not, $\| u_{\lambda,n} \| = n$ for each $n \in \mathbb{N}\backslash\{0\}$. Put $w_{\lambda,n} := u_{\lambda,n}/n$. Then, considering eventually a subsequence, we can assume that

$$w_{\lambda,n} \rightharpoonup w.$$

Moreover, there exists $q \in \mathbb{N}\backslash\{0\}$ such that $u_o \in B_n$ for each $n \ \geq \ q$. Then, for all $n \ \geq \ q, n \in \mathbb{N}$, we put $v := u_o$ in (6) and we get

$$-\lambda d_C^0(u_{\lambda,n}, u_o - u_{\lambda,n}) + \langle Au_{\lambda,n} - f, u_{\lambda,n} - u_o \rangle \ \leq \ 0.$$

By Lemma 2.1,

$$d_C^0(u_{\lambda,n}, u_o - u_{\lambda,n}) \ \leq \ 0,$$

and thus

$$\langle Au_{\lambda,n} - f, u_{\lambda,n} - u_o \rangle \ \leq \ 0,$$

so that

$$R(A, f, u_o) \neq \emptyset,$$

and a contradiction.

We prove that $u_{\lambda,k}$ solves problem (Q_λ). Indeed, for all $y \in X$, there exists $\varepsilon > 0$ such that $u_{\lambda,k} + \varepsilon(y - u_{\lambda,k}) \in B_k$. Take

$$\varepsilon < (k - \| u_{\lambda,k} \|)/ \| y - u_{\lambda,k} \|, \text{ if } y \neq u_{\lambda,k}$$

and

$$\varepsilon = 1, \ \text{if } y = u_{\lambda,k}.$$

If we put $v := u_{\lambda,k} + \varepsilon(y - u_{\lambda,k})$ in (6), we get

$$\varepsilon\lambda d_C^0(u_{\lambda,k}, y - u_{\lambda,k}) + \varepsilon\langle Au_{\lambda,k} - f, y - u_{\lambda,k} \rangle \ \geq \ 0.$$

Since $\varepsilon > 0$ and y is arbitrarely chosen in X, we get our result. ■

We are now able to prove our first existence theorem.

THEOREM 3.1. *Suppose that assumptions (H_1)-(H_3) are satisfied. If*

$$R(A, f, u_o) = \emptyset$$

then (i) problem P has at least one solution and ii) there exist $N \in \mathbb{N}\backslash\{0\}$ such that for all $\theta \geq N, \theta \in \mathbb{N}$, each solution of problem (Q_θ) is also a solution of problem P.

Proof. For each $n \in \mathbb{N}\backslash\{0\}$ we apply Lemma 3.1 to get a sequence $u_n \in X$ such that

$$\langle Au_n - f, v - u_n\rangle + nd_C^0(u_n, v - u_n) \geq 0, \ \forall v \in X. \tag{7}$$

We prove that there exists $M > 0$ such that:

$$\| u_n \| \leq M, \ \forall n \in \mathbb{N}\backslash\{0\}.$$

Assume the contrary. Then by considering eventually a subsequence, we may assume that

$$\| u_n \| \to \infty$$

and

$$w_n = \frac{u_n}{\|u_n\|} \rightharpoonup w.$$

Moreover, if we put $v := u_o$ in (7) then we get

$$\langle Au_n - f, u_n - u_o\rangle - nd_C^0(u_n, u_o - u_n) \leq 0$$

and thus, by using Lemma 2.1, we obtain

$$\langle Au_n - f, u_n - u_o\rangle \leq 0,$$

so that $w \in R(A, f, u_o)$ and a contradiction.

We prove that there exists $N \in \mathbb{N}\backslash\{0\}$ such that $u_\theta \in C$ for all $\theta \geq N, \theta \in \mathbb{N}$. Indeed, if we suppose the contrary, then we can extract a subsequence (again denoted by u_n) such that $u_n \notin C$ and

$$\langle Au_n - f, u_o - u_n\rangle + nd_C^0(u_n, u_o - u_n) \geq 0,$$

which implies by using Lemma 2.1 that

$$\langle Au_n - f, u_o - u_n\rangle \geq nd_C(u_n) + n\rho \geq n\rho.$$

Thus

$$\begin{aligned} n\rho &\leq \| f \|_* \| u_o - u_n \| + \| Au_n \|_* \| u_o - u_n \| \\ &\leq \| f \|_* (M + \| u_o \|) + \| A \| M(M + \| u_o \|) \\ &= \sigma, \end{aligned}$$

with $\sigma := \| f \|_* (M + \| u_o \|) + \| A \| M(M + \| u_o \|)$.

We conclude that:

$$n \leq \sigma/\rho,$$

which is a contradiction for n great enough.

Choose $\theta \geq N$ and put $u := u_\theta$. It is clear that u is a solution of problem P. Indeed, in order to assert that

$$\langle Au - f, v\rangle \geq 0, \forall y \in T_C(u)$$

it suffices to recall that since $u \in C$,

$$y \in T_C(u) \Leftrightarrow d_C^0(u, y) = 0. \quad \blacksquare$$

REMARK 3.1. (i) Various sufficient conditions on A and C guaranteeing the pseudomonotocity of $A + \partial d_C$ can be found in the article of Naniewicz [16] and the book of Naniewicz and Panagiotopoulos [17]. (ii) If A is coercive with respect to u_o, i.e. $\langle Au, u - u_o\rangle / \parallel u \parallel \to +\infty$ as $\parallel u \parallel \to +\infty$, then $R(A, f, u_o)$ is empty. (iii) Proposition 3.2 means that if we can prove the compactness condition requiring the a-compacity of $R(A, f, u_o)$ and the compatibility condition (C) then $R(A, f, u_o)$ is necessarily empty. By using Theorem 3.1 together with Proposition 3.2, we obtain the Corollary 3.1 stated below. (iv) More examples of problems which lead to an empty set of "bad" directions $R(A, f, u_o)$ can be found in the article of Adly *et al.* [1].

COROLLARY 3.1. *Suppose that assumptions (H_1)-(H_3) are satisfied. If*
(i) $R(A, f, u_o)$ *is a-compact,*
(ii) *there exists a nonempty subset* W *of* $X\backslash\{0\}$ *such that* $R(A, f, u_o) \subset W$ *and*

$$(C)\ \underline{r}_{uo,A}(w) > \langle f, w\rangle, \forall w \in W,$$

then problem P has at least one solution

4. A General Existence Theorem

Let us now introduce the following refined set of asymptotic directions:

$$\begin{aligned} R(A, f, C, u_o) := \{&w \in X : \exists u_n \in C, \parallel u_n \parallel \to +\infty, \\ &w_n := u_n \parallel u_n \parallel^{-1} \rightharpoonup w \\ &\text{and } \langle Au_n, u_n - u_o\rangle \leq \langle f, u_n - u_o\rangle\}. \end{aligned}$$

As in Section 3, we say that $R(A, f, C, u_o)$ is *a-compact* if for each $w \in R(A, f, C, u_o)$ the sequences $\{w_n \mid n \in \mathbb{N}\}$ which appear in the definition of this set, are strongly convergent to w, i.e. $w_n \to w$ in X.

THEOREM 4.1. *Suppose that assumptions* (H_1)-(H_3) *are satisfied. If*

$$R(A, f, C, u_o) = \emptyset$$

then problem P has at least one solution.

Proof: Fix $n \in \mathbb{N}\backslash\{0\}$ and let

$$B_k := \{x \in X : \| x \| \leq k\},$$

where $k \in \mathbb{N}\backslash\{0\}$ is chosen great enough so that $u_o \in B_k$.

Let $j \geq k$ be given in $\mathbb{N}$. As in Lemma 3.1, we prove the existence of $u_{n,j} \in B_j$ such that

$$nd_C^0(u_{n,j}, v - u_{n,j}) + \langle Au_{n,j} - f, v - u_{n,j}\rangle \geq 0, \ \forall v \in B_j.$$

We claim that there exists $\theta = \theta(j) \in \mathbb{N}\backslash\{0\}$ such that $u_{\theta,j} \in C$. Indeed, suppose on the contrary that $u_{n,j} \notin C, \forall n \in \mathbb{N}\backslash\{0\}$. Then

$$\langle Au_{n,j} - f, u_o - u_{n,j}\rangle + nd_C^0(u_{n,j}, u_o - u_{n,j}) \geq 0,$$

which implies that

$$\langle Au_{n,j} - f, u_o - u_{n,j}\rangle \geq nd_C(u_{n,j}) + n\rho \geq n\rho.$$

Thus

$$\begin{aligned} n\rho &\leq \| f \|_* \| u_o - u_{n,j} \| + \| Au_{n,j} \|_* \| u_o - u_{n,j} \| \\ &\leq \sigma(j) \end{aligned}$$

with $\sigma(j) := \| f \|_* (j + \| u_o \|) + \| A \| j(j + \| u_o \|)$.

We conclude that for each $n \in \mathbb{N}\backslash\{0\}$

$$n \leq \sigma(j),$$

which is a contradiction.

We prove that there exists $k' \in \mathbb{N}, k' \geq k$ such that $\| u_{\theta(k'),k'} \| < k'$. If not, $\| u_{\theta(i),i} \| = i$ for each $i \in \mathbb{N}, i \geq k$. On relabeling if necessary, the sequence defined by $w_i := w_{\theta(i),i} = u_{\theta(i),i}/i$ satisfies $w_i \rightharpoonup w, u_i := u_{\theta(i),i} \in C$ and

$$\langle Au_i - f, u_i - u_o\rangle \leq 0,$$

which means that $w \in R(A, f, C, u_o)$, a contradiction.

We have

$$\theta(k')d_C^0(u_{k'}, v - u_{k'}) + \langle Au_{k'} - f, v - u_{k'}\rangle \geq 0, \quad \forall v \in B_{k'}. \tag{8}$$

Let $y \in X$, there exists $\varepsilon > 0$ such that

$$u_{k'} + \varepsilon(y - u_{k'}) \in B_{k'}.$$

If we put $v := u_{k'} + \varepsilon(y - u_{k'})$ in (8), we get

$$\theta(k')\varepsilon d_C^0(u_{k'}, y - u_{k'}) + \varepsilon\langle Au_{k'} - f, y - u_{k'}\rangle \geq 0.$$

Since $\varepsilon > 0$, y is arbitrarely chosen in X and $u_{k'} \in C$, we may conclude as in the proof of Theorem 2.1. ∎

COROLLARY 4.1. *Suppose that assumptions (H_1)-(H_3) are satisfied. If*

(i) *$R(A, f, C, u_o)$ is a-compact*

(ii) *there exists a nonempty subset W of $X\backslash\{0\}$ such that*
$R(A, f, C, u_o) \subset W$
and

$$(C)\ \underline{r}_{uo,A}(w) > \langle f, w\rangle,\ \forall w \in W,$$

then problem P has at least one solution.

REMARK 4.1. (i) $R(A, f, C, u_o) \subset R(A, f, u_o)$. (ii) If $R(A, f, u_o)$ is a-compact then $R(A, f, C, u_o)$ is a-compact. (iii) If $R(A, f, C, u_o)$ is a-compact, then $R(A, f, C, u_o) \subset C_\infty \cap R(A, f, u_o)$. (iv) Theorem 4.1 is a more general existence theorem than Theorem 3.1. However, Theorem 3.1 gives us a direct relation between Problem P and its penalization. For instance, if A is a potential operator, i.e. $A = \Phi'$ with $\Phi \in C^1(X, \mathbb{R})$, then a way to solve Problem P is to compute a solution of the optimal program

$$\min\{\Phi(x) + nd_C(u) : x \in X\}$$

for $n \in \mathbb{N}$ great enough.

We are now able to establish two basic results which will be refered to in Section 6. Their corresponding versions in the framework of variational inequalities have been often used in order to study various convex unilateral problems in elasticity (see Fichera [11], Goeleven [12], Kikuchi and Oden [14] and Panagiotopoulos [21]).

THEOREM 4.2. *Let X be a real reflexive Banach space and let $A : X \to X^*$ be a bounded linear operator. We suppose that assumptions (H_1) and (H_2) are satisfied. If A is coercive, i.e. there exists $\alpha > 0$ such that*

$$\langle Au, u\rangle \geq \alpha \parallel u \parallel^2,\ \forall u \in X.$$

Then for each $f \in X^*$, problem P has at least one solution.

Proof. It is clear that

$$\langle Au, u - u_o\rangle / \parallel u \parallel \to +\infty \text{ as } \parallel u \parallel \to +\infty,$$

so that $R(A, f, u_o)$ is empty (Remark 3.1) and thus $R(A, f, C, u_o)$ is empty too. We conclude by application of Theorem 4.1. ■

THEOREM 4.3. *Let X be a real Hilbert space and let $A : X \to X^*$ be a bounded linear operator. We suppose that assumptions (H_1) and (H_2) are satisfied. If*

(i) *A is semicoercive, i.e. there exists* $\alpha > 0$ *such that*

$$\langle Au, u\rangle \ \geq \ \alpha \parallel Pu \parallel^2, \ \forall u \in X,$$

with $P = I - Q$, *where I denotes the identity mapping and Q denotes the orthogonal projection of X onto* $Ker(A + A^*)$(A^* *is the adjoint operator of A);*

(ii) $\dim\{Ker(A + A^*)\} < +\infty$;

(iii) $u_o \in KerA$;

(iv) $\langle f, w\rangle \ < 0, \ \forall w \in C_\infty \cap (KerA + A^*)\backslash\{0\}$.

Then problem P has at least one solution.

Proof. We will prove that all assumptions of Corollary 4.1 are satisfied. We claim that A satisfies the S^+property. Indeed, let $\{u_n \mid n \in \mathbb{N}\}$ be a sequence such that

$$u_n \rightharpoonup u \text{ in } X,$$

and

$$\limsup\langle Au_n, u_n - u\rangle \ \leq \ 0.$$

We have

$$\begin{aligned}\alpha. \limsup \parallel Pu_n - Pu \parallel^2 \ &\leq \ \limsup\langle Au_n - Au, u_n - u\rangle \\ &\leq \ \limsup\langle Au_n, u_n - u\rangle \\ &\quad + \limsup\langle Au, u - u_n\rangle \\ &\leq \ 0,\end{aligned}$$

and thus $Pu_n \rightarrow Pu$. Moreover, Q is bounded linear and thus weakly continuous. Therefore $Qu_n \rightarrow Qu$ since $\dim\{Ker(A + A^*)\} < +\infty$. Thus

$$u_n = Pu_n + Qu_n \rightarrow Pu + Qu = u.$$

It is also clear that all other assumptions required by Proposition 3.1 are also satisfied. Thus (Remarks 4.1) $R(A, f, C, u_o)$ is a-compact and $R(A, f, C, u_o) \subset W$, where

$$W = C_\infty \cap \{x \in X : \langle Ax, x\rangle = 0\}\backslash\{0\} = C_\infty \cap (Ker(A + A^*))\backslash\{0\}.$$

We have

$$\langle A(tx), tx - u_o\rangle/t \ \geq \ -\langle Ax, u_o\rangle, \ \forall x \in X.$$

Thus

$$\underline{r}_{uo,A}(w) \ \geq \ -\langle Aw, u_o\rangle = 0, \ \forall w \in W, \tag{9}$$

so that assumption (iv) together with (9) imply condition (C) of Corollary 4.1. ■

REMARK 4.2. (i) If A is semicoercive, C is a subset of X which can be represented as the union of a finite collection of nonempty closed convex subsets $C_j(j = 1, ..., N)$ of X, i.e. $C = \cup_{j=1}^{N} C_j$ and if $u_o \in int\{\cap_{j=1}^{N} C_j\}$, then assumptions ($H_1$) and ($H_2$) are satisfied. Indeed, we known that A is maximal monotone and satisfies the S^+-property so that we may conclude by using Proposition 2.1. (ii) If A is symmetric, then $W = C_\infty \cap (KerA)\backslash\{0\}$ and $\underline{r}_{uo,A}(w) \ \geq \ 0, \forall w \in W$, so that assumption (iii) on u_o is not necessary.

5. A Nonlinear Perturbation of Elliptic Linear Hemivariational Inequalities Where the Constraints Are Defined by the Finite Union of Closed Convex Sets

Let us consider the problem: Find $u \in C$ such that

$$\int_\Omega \nabla u.\nabla v dx + \int_\Omega f(x,u).v dx - \int_\Omega g(x).v dx \ \geq \ 0, \ \forall v \in T_C(u), \tag{10}$$

where C can be represented as the union of a finite collection of closed convex subsets $C_j(j = 1, ..., N)$ of $H^1(\Omega)$ where Ω is an open bounded of class C^1 subset of $\mathbb{R}^p(p \in \mathbb{N}, p \ \geq \ 1)$. Let g be a fixed element of $L^2(\Omega)$ and let f be a caratheodory function satisfying conditions $(f_1) - (f_2)$ of Proposition 3.3. In this case, the functions

$$f_\infty(x) := \liminf\{f(x,u) : u \to +\infty\}$$
$$f^\infty(x) := \limsup\{f(x,u) : u \to -\infty\}$$

are well defined (see Brézis and Nirenberg [5]).

THEOREM 5.1. *If*

(i) $0 \in \cap_{i=1}^{N} int\{C_i\}$

(ii) $\int_\Omega f^\infty(x)dx < \int_\Omega g(x)dx < \int_\Omega f_\infty(x)dx$,

then problem (10) has at least one solution

Proof. Set

$$\langle Au, v\rangle := \int_\Omega \nabla u.\nabla v dx,$$
$$\langle Bu, v\rangle := \int_\Omega f(x,u).v dx,$$

and

$$\langle f, v\rangle := \int_\Omega g.v dx.$$

We know that A is a bounded maximal monotone operator satisfying the S^+-property. The distance function of C is expressed as the pointwise minimum of convex functions d_n where $d_n : H^1(\Omega) \to \mathbb{R}$ denotes the distance function of C_n, i.e. $d_C = \min\{d_1, ..., d_N\}$. Thus, by Lemma 2.1, $A + \lambda\partial d_C(\lambda \geq 0)$ is pseudomonotone. With condition (f_1) on f, the corresponding operator B is strongly continuous (i.e. $u_n \rightharpoonup u \Rightarrow Bu_n \to Bu$) and thus $A + B + \lambda\partial d_C$ is pseudomonotone too. Moreover, it is clear that $A + B$ is bounded. Assumption (i) implies that C is star-shaped with respect to a certain ball with the origin in 0. By Proposition 3.3, $R(A+B, f, 0)$ is a-compact and $R(A+B, f, 0) \subset KerA\backslash\{0\} = \mathbb{R}\backslash\{0\}$.

It remains to prove that

$$\underline{r}_{A+B,0}(c) > \langle f, c\rangle, \ \forall c \in \mathbb{R}\backslash\{0\}. \tag{11}$$

By ([5]; Proposition ii.4),

$$\underline{r}_{B,0}(w) \geq \int_{\Omega^+(w)} f_\infty(x).w \, dx + \int_{\Omega^-(w)} f^\infty(x).w \, dx$$

with

$$\Omega^+(w) := \{x \in \Omega \mid w(x) > 0\},$$

and

$$\Omega^-(w) := \{x \in \Omega \mid w(x) < 0\}.$$

Moreover

$$\underline{r}_{A+B,0}(w) \geq \underline{r}_{A,0}(w) + \underline{r}_{B,0}(w) \geq \underline{r}_{B,0}(w),$$

and condition (4.2) will be satisfied if

$$\int_{\Omega^+(c)} f_\infty(x).c \, dx + \int_{\Omega^-(c)} f^\infty(x).c \, dx > \int_\Omega g(x).c \, dx, \ \forall c \in \mathbb{R}\backslash\{0\},$$

which is equivalent to

$$\int_\Omega f^\infty(x) \, dx < \int_\Omega g(x) \, dx < \int_\Omega f_\infty(x) \, dx. \qquad \blacksquare$$

6. A Nonconvex Unilateral Contact Problem in Elasticity

Let Ω be a body identified as a bounded open connected subset of $\mathbb{R}^3$ referred to a coordinate system $\{0, x_1, x_2, x_3\}$ and Γ be the body's surface supposed to be regular (i.e. Γ is an hypersurface of class $C^m(m \geq 1)$ and Ω is located on one side of Γ). It is assumed that Ω is subjected to a body force density F. Surface tractions t are applied to a portion Σ of Γ. The body Ω is assumed to be fixed along an open subset Γ_U of Γ (possibly empty). We suppose that $\Gamma_U \cap \Sigma = \emptyset$.

Let $\sigma = \{\sigma_{ij}\}$ be the stress tensor and let $n = \{n_i\}$ be the outward unit normal vector on Γ. We denote by $S = \{S_i\}$ the stress vector on Γ, i.e. $S_i = \sigma_{ij}.n_j$.

Let u denotes the displacement field of the body. We consider the case of infinitesimal deformations of the body and we supppose that the body's material is characterized by a Cauchy elastic law, i.e. $\sigma_{ij} = C_{ijkl}.\varepsilon_{ij}$ where $\varepsilon = \{\varepsilon_{ij}(u)\}$ is the strain tensor and $C = \{C_{ijkl}(x)\}$ is the linear-elasticity tensor. The elasticity tensor $C \in [L^\infty(\Omega)]^{81}$ is supposed to satisfy the classical symmetry properties:

$$C_{ijkl}(x) = C_{ijkl}(x) = C_{ijkl}(x),$$

and the ellipticity property

$$C_{ijkl}(x)\zeta_{ij}\zeta_{kl} \;\geq\; m\zeta_{ij}\zeta_{kl}\;(m > 0),\; \forall x \in \Omega,$$

and for all 3×3 symmetric matrices ζ. The displacement field u satisfies the following system of equations:

(1) $\frac{-\partial\sigma_{ij}}{\partial x_j} = F_i$ in Ω,
(2) $S_i = t_i$ on Σ,
(3) $u = 0$ on Γ_U.

Let X be the Hilbert space defined by

$$X := \{v \in [H^1(\Omega)]^3 : v = 0 \text{ a.e. on } \Gamma_U\}.$$

Then, as a weak formulation of system (1)–(3) we consider the variational equality

$$u \in X : a(u,v) = \langle f, v\rangle \text{ for all } v \in X, \tag{12}$$

where $a(u,v)$ is the bilinear continuous symmetric form

$$a(u,v) = \int_\Omega C_{ijhk}\varepsilon_{ij}(u)\varepsilon_{hk}(v)\mathrm{d}\Omega,$$

and $\langle v, f\rangle$ is the linear continuous form

$$\langle v, f\rangle = \int_\Omega F_i v_i \mathrm{d}\Omega + \int_\Sigma t_i v_i \mathrm{d}s$$

with $F \in [L^2(\Omega)]^3$ and $t \in [L^2(\Sigma)]^3$.

Let us now assume that constraints on the displacement can be represented by a closed subset C of X. We introduce a reaction force $R \in X^*$ in order to describe the action of the constraints on the body and we assume a normal contact law: $u \in C, -R \in N_C(u)$. In this case the system is described by

$$a(u,v) = \langle f + R, v\rangle,\; \forall v \in X, \tag{13}$$

$$-R \in N_C(u), \quad u \in C. \tag{14}$$

Combining (13) and (14) we obtain the hemivariational inequality

$$u \in C : a(u,v) \;\geq\; \langle f, v\rangle,\; \forall v \in T_C(u). \tag{15}$$

We refer to ([17], section 7.16) for similar considerations and other problems in elasticity which lead to a model like (15).

Let $A : X \to X^*$ be the bounded linear and symmetric operator defined by

$$\langle Au, v \rangle = a(u, v), \ \forall u, v \in X.$$

It is known that if $\mu(\Gamma_U) > 0$ then A is coercive. However, if $\Gamma_U = \emptyset$ then A is semicoercive and $KerA = \{v \in X : v(x) = a \wedge x + b, a, b \in \mathbb{R}^3\}$, where $\wedge$ denotes the vector product in $\mathbb{R}^3$.

THEOREM 6.1. *Suppose that the set C is defined as the union of a finite collection of closed convex subsets $C_j (j = 1, ..., N)$ of X. We assume that $int\{\cap_{j=1}^N Q_j\} \neq \emptyset$. Then (i) if $\mu(\Gamma_U) > 0$ then problem (15) has at least one solution for each $f \in X^*$, and (ii) if $\Gamma_U = \emptyset$ then problem (15) has at least one solution for each $f \in X^*$ satisfying the inequality*

$$\langle f, w \rangle < 0, \forall w \in C_\infty \cap Ker(A) \backslash \{0\}.$$

Proof. (i) By Theorem 4.2 and Remark 4.2 (i). (ii) By Theorem 4.3 and Remark 4.2. ∎

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